FMI CDBS 2019-2020 - Homework 2

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Let E be a real Banach space.

Definition 1. ([Phe93, Example 2.26]) We define the duality mapping

$$D : E \rightrightarrows E^*,$$
$$D(x) \coloneqq \{x^* \in E^* \colon \ \|x\| = \|x^*\| \ \text{ and } \ \langle x^*, x\rangle = \|x^*\| \, \|x\|\}.$$

Note. We will only use this mapping for unit vectors, so we may as well consider its restriction to the unit spheres, where

$$D' : S_E \rightrightarrows S_{E^*},$$
$$D'(x) \coloneqq \{x^* \in S_{E^*} \colon \ \langle x^*, x\rangle = 1\}.$$

Definition 2. ([Phe93, Definition 2.36]) The norm $\|\cdot\|$ on E is said to be

a) Strictly convex (or rotund) if there is no line segments in the unit sphere S_E.

b) Smooth if for each $x \in S_E$ the duality mapping is single-valued.

I decided to prove the following since it was not obvious to me

Lemma 3. *For every point $x \in E$, the set $D(x)$ is nonempty.*

Proof. Fix $x \in E$ and consider the one-dimensional subspace

$$\operatorname{span}\{x\} = \{\lambda x \colon \lambda \in \mathbb{R}\}.$$

Define $\xi : \operatorname{span}\{x\} \to \mathbb{R}$ by $\xi(\lambda x) = \lambda \|x\|^2$.

The functional ξ is linear and, since it acts on a finite-dimensional space, it is also continuous. The norm of ξ is

$$\|\xi\| = \max\left\{ \left\langle \xi, \frac{x}{\|x\|} \right\rangle, \left\langle \xi, -\frac{x}{\|x\|} \right\rangle \right\} = \max\{\|x\|, -\|x\|\} = \|x\|.$$

The Hahn-Banach theorem allows us to extend ξ to a continuous linear functional $x^* \in E^*$ such that $\|x^*\| = \|\xi\| = \|x\|$ and $\langle x^*, x\rangle = \langle \xi, x\rangle = \|x\|^2 = \|x\| \, \|x^*\|$. Thus $x^* \in D(x)$ and $D(x)$ is nonempty. $\square$

Exercise 1. ([Phe93, Exercise 2.37]) Prove that:

a) If the norm in E is such that its dual norm in E^* is rotund (resp. smooth), then it is itself smooth (resp. rotund).

b) The norm in E is rotund iff every convex subset of E has at most one point of least norm.

c) Norms in Hilbert spaces are both smooth and rotund, but the norms in c_0 and l^1 are neither.

d) The norm in E is strictly convex iff $\|x + y\| < \|x\| + \|y\|$ whenever x and y are linearly independent.

Proof. 1. 1) First, let the dual norm $\|\cdot\|^*$ be rotund and assume that $\|\cdot\|$ is not smooth.

Fix $x \in S_E$. Since $D(x)$ is nonempty (by lemma 3) and since $\|\cdot\|$ is not smooth, then there exist two different functionals $x^*, y^* \in D(x)$, such that

$$\langle x^*, x\rangle = \langle y^*, x\rangle = 1.$$

We will show that the segment $[x^*, y^*]$ is contained in S_{E^*}, i.e. that the dual norm is not rotund.

Fix any $t \in (0, 1)$ and define $z^* := tx^* + (1-t)y^*$. We only need to show that $\|z^*\| = 1$.

By the triangle inequality, we have

$$\|z^*\| = \|tx^* + (1-t)y^*\| \le t\,\|x^*\| + (1-t)\,\|y^*\| = t + (1-t) = 1.$$

For the reverse inequality, note that

$$\|z^*\| \ge \langle z^*, x\rangle = t\,\langle x^*, x\rangle + (1-t)\,\langle y^*, x\rangle = t + (1-t) = 1,$$

thus $\|z^*\| = 1$. Hence $[x^*, y^*]$ is contained in S_{E^*} and the dual space is not smooth. The obtained contradiction proves that the norm in E is rotund.

2) Now let the dual norm $\|\cdot\|^*$ be smooth and assume that $\|\cdot\|$ is not rotund. Then there exist points $x, y \in S_E$ such that the while segment $[x, y]$ is contained in S_E.

Fix $t \in (0, 1)$ and define $z := tx + (1-t)y \in S_X$. Denote by $J : E \to E^{**}$ the canonical embedding into the double-dual. By lemma 3, there exists a functional $z^* \in E^*$, such that

$$\langle J(z), z^*\rangle = \langle z^*, z\rangle = 1.$$

Because the dual norm $\|\cdot\|^*$ is smooth, we cannot have $\langle J(x), z^* \rangle = \langle z^*, x \rangle = 1$ or $\langle J(y), z^* \rangle = \langle z^*, y \rangle = 1$ and since $\|z^*\| = 1$, necessarily

$$\langle z^*, x \rangle < 1 \text{ and } \langle z^*, y \rangle < 1.$$

If follows that

$$1 = \langle z^*, z \rangle = t \langle z^*, x \rangle + (1-t) \langle z^*, y \rangle < t + (1-t) = 1,$$

which is a contradiction. Hence $\|\cdot\|$ is rotund.

2. ($\implies$) Let the norm in E be rotund and let $C \subseteq E$ be a (potentially empty) convex set. We will prove that C contains at most one point of least norm.

   If C is empty or otherwise contains no element of least norm, trivially contains at most one point of least norm.

   Now let C contain at least one element $x \in C$ of least norm. Assume that $y \in C$ is another element of least norm. Necessarily $\|x\| = \|y\|$.

   Fix $t \in (0,1)$ and define $z := tx + (1-t)y$. Since C is convex, it contains z. Since x and y are elements of least norm, we have $\|z\| \geq \|x\|$. By the triangle inequality,

$$\|z\| = \|tx + (1-t)y\| \leq t \|x\| + (1-t) \|y\| = \|x\|,$$

   thus $\|z\| = \|x\|$.

   This implies that the entire segment $[x, y]$ are elements of least norm in C. Hence the segment $[x, y]$ is contained in the sphere $\|x\| S_E$, which contradicts the rotundity of the norm $\|\cdot\|$.

   Hence C contains at most one element of least norm.

   ($\impliedby$) Let every convex set $C \subseteq E$ have at most one element of least norm.

   Assume that the norm $\|\cdot\|$ is not rotund. Then the unit sphere S_E contains a line segment $[x, y], x \neq y$. The set $[x, y]$ is compact and, by the Weierstrass extreme value theorem, the norm attains its minimum on the segment in a point $z \in [x, y]$. Since the segment is also convex and we assumed that convex sets have at most one element of least norm, it follows that this element z is unique.

   Then for any point $s \in [x, y], s \neq z$, we have $\|s\| > \|z\| = 1$, thus s cannot be an element of the unit sphere. The obtained contradiction shows that the norm $\|\cdot\|$ is rotund.

3. 1) Let E be a Hilbert space, i.e. the norm is generated by an inner product and, due to Riesz's theorem, we identify the space E with its continuous dual E^*.

   To prove that E is rotund, choose $x, y \in S_E, x \neq y$. We will show that the segment $[x, y]$ is not contained in S_E.

   If x and y are linearly dependent, necessarily $y = -x$ and all non-trivial convex combinations of x and y are contained in the open unit ball, hence $[x, y] \not\subseteq S_E$.

Not let x and y be linearly independent. By the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$\langle x, y \rangle \leq |\langle x, y \rangle| < \|x\| \, \|y\| = 1. \tag{1}$$

Fix $t \in (0,1)$ and let $z \coloneqq tx + (1-t)y$. We will show that $z \notin S_E$. Indeed,

$$\begin{aligned} \|z\|^2 = \langle z, z \rangle = t^2 \|x\|^2 + t(1-t) \langle x, y \rangle + (1-t)t \langle y, x \rangle + (1-t)^2 \|y\|^2 = \\ = t^2 + (1-t)^2 + 2t(1-t) \langle x, y \rangle < \\ \overset{(1)}{<} t^2 + (1-t)^2 + 2t(1-t) = \\ = t^2 + 1 - 2t + t^2 + 2t - t^2 = 1. \end{aligned}$$

Thus $\|z\|^2 < 1$ and $\|z\| < 1$ and $z \notin S_E$.

In both cases, no interior point of the segment $[x, y]$ is contained in S_E, hence the norm in E is rotund.

Since we identify E with its dual, the norm in E^* is also rotund and by a), the norm in E is also smooth.

2) Consider the space c_0 of all real sequences that converge to zero equipped with the uniform norm

$$\|x\|_{c_0} \coloneqq \sup_i |x_i| \, .$$

Note that the dual space of c_0 is (isometrically isomorphic to) the space l^1 of absolutely summable sequences with norm

$$\|x\|_{l^1} \coloneqq \sum_i |x_i| \, .$$

Let $\{e_n\}_{n=1}^{\infty}$ be the canonical basis of c_0, i.e. the coordinates $e_n^{(i)}$ of e_n are given by the Dirac delta function, $e_n^{(i)} \coloneqq \delta_{i,n}$.

For every natural $n \geq 1$, define x_n to be the same as e_n except that the first coordinate of x_n is always 1.

The corresponding norms of e_n are all equal to 1 and the norms of x_n are

$$\|x_n\|_{c_0} = 1 \qquad\qquad \|x_n\|_{l^1} = 2.$$

For every n we have

$$\langle e_1, x_n \rangle = \langle e_n, x_n \rangle = 1,$$

hence $J_{c_0}(x_n)$ has at least two elements e_1 and e_n and the norm in c_0 is not smooth.

Given that $\{x_1, x_2, \ldots\} \subseteq S_{c_0}$, consider the convex combinations of x_2 and x_3:

$$tx_2 + (1-t)x_3 = (1, t, (1-t), 0, 0, \ldots).$$

Evidently $tx_2 + (1-t)x_3 \in S_{c_0}$ for every $t \in (0,1)$, hence the norm in c_0 is not rotund.

The contrapositions to the statements in a) say that if E is not rotund (resp. smooth), then the dual space E^* is not smooth (resp. rotund). Thus l^1 is neither smooth or rotund as the dual of c_0.

4. We will prove that E is rotund if and only if

$$\|x+y\| = \|x\| + \|y\| \implies x \text{ and } y \text{ are linearly dependent.} \tag{2}$$

$(\implies)$ Let E be rotund let $x, y \in E$ be distinct vectors such that

$$\|x+y\| = \|x\| + \|y\|. \tag{3}$$

If either of them is the zero vector, then they are trivially linearly dependent.

Assume that both x and y are nonzero and define

$$\xi := \frac{x}{\|x\|} \qquad \eta := \frac{y}{\|y\|} \qquad t := \frac{\|x\|}{\|x+y\|}$$

Equation (3) implies that

$$1 - t = 1 - \frac{\|x\|}{\|x+y\|} = \frac{\|x+y\| - \|x\|}{\|x+y\|} = \frac{\|y\|}{\|x+y\|}.$$

Since both ξ and η are in S_E, by rotundity, their convex combination

$$\nu := t\xi + (1-t)\eta$$

should not be contained in S_E unless $\xi = \eta$.

Calculating the norm, we obtain

$$\begin{aligned} \|\nu\| &= \|t\xi + (1-t)\eta\| = \\ &= \left\| \frac{\|x\|\,\xi}{\|x+y\|} + \frac{\|y\|\,\eta}{\|x+y\|} \right\| = \\ &= \left\| \frac{x+y}{\|x+y\|} \right\| = 1, \end{aligned}$$

hence $\nu \in S_E$. Thus $\xi = \eta$ and $x = \frac{\|x\|}{\|y\|} y$, so x and y are linearly dependent.

($\Longleftarrow$) Let eq. (2) hold and fix $x, y \in S_E, t \in (0,1)$. Define $z \coloneqq tx + (1-t)y$. First, assume that the vectors tx and $(1-t)y$ satisfy the left part of eq. (2), i.e.

$$\|z\| = \|tx + (1-t)y\| = t\,\|x\| + (1-t)\,\|y\| = 1.$$

This does not refute rotundity since x and y are not necessarily distinct. It follows from eq. (2) that tx and $(1-t)y$ are linearly dependent, hence x and y are also linearly dependent. Since x and y both have unit norm, either $y = x$ or $y = -x$.

If we assume that $y = -x$, then

$$\|z\| = \|tx + (1-t)y\| = (2t-1)\,\|x\| = 2t-1,$$

which is only possible if $t = 1$ since $\|z\| = 1$. But t is strictly less than 1.

Hence $y \neq -x$ and the only remaining possibility is that $y = x$.

Now assume that the vectors tx and $(1-t)y$ do not satisfy the left part of eq. (2). This implies $\|z\| < 1$. Thus x and y are necessarily distinct, but z is not contained in the unit sphere and the segment $[x, y]$ is not contained in S_E.

We have shown that $x, y \in S_E$ implies that either $y = x$ or that the segment $[x, y]$ is not contained in S_E, thus the norm in E is rotund.

$\square$

References

[Phe93] Robert Phelps. Convex functions, monotone operators, and differentiability. Springer-Verlag, 1993. ISBN: 0387567151.